Fundamental gravitational limitations to quantum computing

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Lloyd [1] has considered the ultimate limitations physics places on quantum computers. He concludes in particular that for an "ultimate laptop" (a computer of one liter of volume and one kilogram of mass) the maximum number of operations per second is bounded by 10^{51}. The limit is derived considering ordinary quantum mechanics. Here we consider additional limits that are placed by quantum gravity ideas, namely the use of a relational notion of time [2] and fundamental gravitational limits [3] that exist on time measurements. We then particularize for the case of an ultimate laptop and show that the maximum number of operations is further constrained to 10^{47} per second.

Ordinary quantum mechanics is an approximate theory. It is based on the existence of a universal and perfectly classical time. Such a notion is not compatible with our understanding of relativity and gravitation. The latter requires time to emerge as a relational variable [4]. The relation is between the object being studied and what we choose as a clock to study it. Quantum mechanics formulated in terms of a relational time differs from ordinary quantum mechanics [5]. The "clock" is now a quantum variable with quantum (and thermal) fluctuations. In particular quantum states lose coherence over time due to the fact that any clock in nature is imperfect [6]. One can find a fundamental level of loss of coherence that is inescapable if one considers the most accurate clock that the laws of physics allow us to construct. Such clocks were considered by Ng and Van Dam [3].

What we would like to argue in this paper is that the effect of loss of coherence due to the lack of a perfect clock in classical mechanics imposes limitations on the ultimate performance of quantum computers. Although the effects are derived from gravitation and therefore are expected to be very small, they are large enough to impose limits on quantum computation several orders of magnitude smaller than those found by in reference [1] considering ordinary quantum mechanics.

In that paper it was noted that for a quantum computer of a finite energy E, one had a finite number of operations that were possible per second. This is a consequence of the Margolus–Levitin [7] theorem that states that the time it takes for a quantum state to evolve into an orthogonal state is $\delta t \geq \pi/(2E)$, where E is the expectation value of the energy and throughout this paper we choose units so $\hbar = 1$. Therefore a system with energy E can carry out a maximum of $2E/\pi$ logical operations per second. This bound is independent on the computer being parallel or serial. Therefore for a computer weighing one kilogram utilizing all its mass-energy resources and henceforth operating at the limits of speed (an "ultimate laptop"), the maximum number of operations per second is $n \sim 10^{51}$.

The above reasoning is based on the premise that logical operations are implemented quantum mechanically by exactly unitary evolutions. But as we argued above, in quantum mechanics with real clocks evolution is never perfectly unitary. Therefore there is the possibility that the above bounds on quantum computation may have to be revised.

The origin of the lack of unitarity is the fact that definite statistical predictions are only possible by repeating an experiment. If one uses a real clock, which has thermal and quantum fluctuations, each experimental run will correspond to a different value of the evolution parameter. The statistical prediction will therefore correspond to an average over several intervals, and therefore its evolution cannot be unitary. This effect has been observed in the Rabi oscillations describing the exchange of excitations between atoms and field [8].

There is growing evidence that there exists a fundamental limit to how accurate a clock can be. Arguments for the limit involve simple relations derived from basic quantum mechanical principles and gravitational physics. In their original incarnation the arguments were based on the fact that more accuracy requires the clock to be more massive [9] but this in turn is limited in the gravitational context since large accumulations of mass in small regions turn the region into a black hole [3]. This yields an ultimate accuracy for a clock of the order of $\delta t \geq t_{\rm P}^{2/3} t^{1/3}$. where $t_{\rm P}$ is Planck's length. Several other arguments yield the same limit [10]. Attractively, the limit also leads naturally to the Bekenstein bound [11, 12].

The fact that there is a limit to how accurate a clock can be, coupled with the observation that quantum mechanics with real clocks fails to be unitary leads naturally to an estimation on the rate of fundamental loss of coherence of quantum states. In the limit of highly accurate clocks, the evolution equation of the density matrix of states takes the approximate form [2, 5],

$$\frac{\partial \rho}{\partial t} = i[\rho, H] - \sigma[[\rho, H], H], \tag{1}$$

where σ is related to the rate at which the uncertainty in the clock variable grows. For $\sigma = 0$ one would recover ordinary quantum mechanics. However, as we argued above there is a lower bound on the value of σ motivated from the best possible clock one can build (a black hole), which turns out to be [2],

$$\sigma(t) = \frac{t_P}{36}\left(\frac{t_P}{T_{\max} - t}\right)^{1/3}, \tag{2}$$

where $T_{\max}$ is the interval of time in which one is interested in studying the system.

To study the influence of this effect on quantum computers, we start by recalling the argument by Margolus and Levitin [7] who showed that to transition from a generic quantum state $|\psi_0>$ to an orthogonal quantum state $|\psi_t>$ takes a minimum amount of time $t \geq \pi/(2E)$ where E is the expectation value of energy of the system (assuming the ground state has zero energy).

As we stated, when one has loss of coherence a state never evolves completely into an orthogonal one. So if one starts with a density matrix initially of the form $\rho^0_{mn} = c_m c_n^*$, it will evolve into

$$\rho^t_{mn} = c_m e^{-i\omega_m t} c_n^* e^{i\omega_n t} e^{-(\omega_m-\omega_n)^2 t_{\rm P}^{4/3} t^{2/3}}, \tag{3}$$

and the transition amplitude will be

$$\mathrm{Tr}\left(\rho^t_{mn}\rho^0_{mn}\right) = \sum_{m,n} |c_m|^2 |c_n|^2 e^{i(\omega_m-\omega_n)t} e^{-(\omega_m-\omega_n)^2 t_{\rm P}^{4/3} t^{2/3}}. \tag{4}$$

For instance, let us consider a NOT gate in a quantum computer, that is a gate that takes a single binary input X and returns the output 1 if X=0 and 0 if X=1. Taking as initial state $|\psi_0> = (|E_0> + |E_1>)/\sqrt{2}$ and final state $|\psi_1> = (|E_0> - |E_1>)/\sqrt{2}$ we will have for the action of the gate after a time $t \sim \pi/(2E)$,

$$|\psi_0><\psi_0| \rightarrow (1-\epsilon)|\psi_1><\psi_1| + \epsilon|\psi_0><\psi_0|, \tag{5}$$

with $\epsilon = 4t_{\rm P}^{4/3} t^{2/3} E^2$. Therefore one sees that the loss of coherence induces error in the quantum computation and the probability of the error is given by ϵ per logical operation. This effect forces to include error correction [13]. Error correction also has to be introduced to compensate for environmental effects, but we are not considering these here, since we are seeking a fundamental limit that is inescapable even if one eliminates all environmental effects. There are fundamental limits on how much error correction can be introduced in a quantum computer. Although there may be many mechanisms for error correction, they can all be pictured as the computer communicating its state to an "error correcting device". Such communication cannot occur faster than the speed of light. This limits the rate at which errors can be corrected and therefore implies a maximum tolerable amount of errors per operation. The rate at which information can be extracted from a computer with L bits stored and size R is given by Lc/R and c is the speed of light. If we divide by the number of operations per second n we get the maximum tolerable level of errors per operation for the machine $\epsilon_{\max} \sim Lc/(nR)$.

The extension of the Margolus–Levitin result to the case in which we have fundamental decoherence (as we sketched above) establishes a bound on the speed of operations that is state-dependent. The bound is saturated when the computer is operating in "serial mode", i.e. all its mass-energy resources and stored bits, E, L, are used per logical operation and therefore is in a state that is a superposition of states that are widely separated in energy. In such states the computer achieves a very fast "clock rate" $t_{\rm step} \sim 1/E$. However, it can only carry out a few operations per clock cycle since its bits are highly entangled. In other words, most of the stored bits are used to perform a single quantum operation at a high speed. For these types of states the decoherence effect we are discussing in this paper is maximum (recall that the effect goes as the energy squared). On the other hand, if one considers states that are in "parallel mode", that is, where a considerable number of bits are not entangled and each perform independent quantum operations, the energy differences are smaller and the decoherence effect gets weaker. Nonetheless it is still dominant for the case of an ultimate laptop as we shall see below.

Let us compute the decoherence error one would introduce for a 1kg quantum computer using a total mass-energy of $E = mc^2 \sim 10^{16} J$ in serial model. This turns out to be,

$$t_{\rm P}^{4/3}\left(\frac{1}{E}\right)^{2/3} E^2 \sim 10^9, \tag{6}$$

which is remarkably large. We are therefore led to conclude that such a quantum computer cannot utilize all its resources to compute in serial mode. As it was pointed out by Lloyd [1] an ultimate laptop working at its maximum of capacity would have a degree of parallelization (d_p), defined in Ref. [1] to be roughly $d_p \sim 1/\epsilon_{max}$, of the order of

$d_p \sim 10^{10}$ for the ultimate laptop. It is therefore crucial to extend our effect to the case of parallel computation. It is easy to see that the only difference with the calculation in (6) is that now the energy is redistributed amongst L/d_p parallel qubits and therefore the energy per gate goes down to E/d_p (the case $d_p = 1$ will account naturally for the serial mode). Similarly to what happens in the serial case, we will be also led to conclude that an ultimate laptop cannot utilize all its resources without running into an unavoidable error crash. In order to see this let us particularize the bound we previously obtained for the error rate for the case of a quantum computer of size R and L bits stored,

$$t_{\mathrm{P}}^{4/3}\left(\frac{d_p}{E_{\mathrm{eff}}}\right)^{2/3}\left(\frac{E_{\mathrm{eff}}}{d_p}\right)^{2} \leq \frac{cL}{nR}, \tag{7}$$

where E_{eff} is the *effective* energy the quantum computer can actually invest with a degree of parallelization d_p. Given now that the number of operations per second in the later is bounded by $n < E_{\mathrm{eff}}$, we have that,

$$n \leq \left(\frac{1}{t_{\mathrm{P}}}\right)^{4/7}\left(\frac{cL}{R}\right)^{3/7} d_p^{4/7} \sim 10^{47}\mathrm{op/s}. \tag{8}$$

This expression is general for a quantum computer of L bits and characteristic size R operating with degree of parallelization d_p, and the numerical estimate is obtained from Ref. [1] where $L \sim 10^{31}$, $R \sim 0.1m$ and $d_p \sim 10^{10}$ for an ultimate laptop with volume one liter (this is also related to the maximum entropy that can be contained in the volume [14]). In addition, (8) also leads us to conclude that a 1 kg quantum computer working at serial mode ($d_p = 1$) can not perform more than 10^{42} op/s.

These bounds, though large, are three and nine orders of magnitude more stringent for parallel and serial computation respectively, than the one found by Lloyd (that yields the same bound for both cases).

Finally, if one is interested in miniaturization, one may wish to consider what are the limits on the most compact computer one can manufacture. Such a computer is a black hole, as argued by Ng and Lloyd [10, 15]. In this case the maximum number of bits is given by the formula of Bekenstein. A similar calculation to the one above leads to an estimate of $n \leq (M/M_{\mathrm{Planck}})^{3/7}/t_{\mathrm{Planck}}$ that for a kilogram mass black hole is approximately 10^{47}op/s. So the black hole computer faces the same limitations as the "ultimate laptop" due to the effect we consider.

Finally, let us add that if one wishes to consider an "Avogadro computer", a more realistic sort of computer in which qubits consist of atomic nuclei, the maximum number of operations per second is reduced to 10^{39} working in serial mode (this is largely due to the fact that the number of qubits L that appears in the above formulae is reduced to $L \sim 10^{25}$), a bound slightly tighter than the one implied by the Margolus–Levitin theorem (10^{40}). If the computer operates in parallel one can reach the latter limit.

Summarizing, we have found that formulating quantum mechanics properly in terms of realistic clocks yields fundamental limitations on quantum computers that are more stringent than other bounds of similar nature found up to present.

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